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1972 J. Phys. A: Gen. Phys. 5 L49

(http://iopscience.iop.org/0022-3689/5/6/001)

J. Phys. A: Gen. Phys., Vol. 5, June 1972. Printed in Great Britain

LETTER TO THE EDITOR

The effect of heterodyne detection on the statistical accuracy of optical linewidth measurements

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MS received 14 April 1972

Abstract. It is shown that an order of magnitude gain in the statistical accuracy of optical linewidths measured by digital autocorrelation of photon-counting fluctuations may be achieved by heterodyne detection in the weak signal limit.

There has been a good deal of interest recently in the potential accuracy of information extracted from an optical signal by intensity fluctuation spectroscopy (IFS). Both the optimum design of a particular signal processing system and the optimum system have been discussed assuming a given input signal strength and experiment time (Jakeman *et al* 1971, Degiorgio and Lastovka 1971, Kelly 1971). However, in addition to the system of post-detection analysis there exists the possibility of pre-detection processing by a heterodyne technique, that is, by mixing the signal of interest (spectral peak frequency ω_s) with the coherent field from a local oscillator source (frequency ω_c) on the photocathode of the detector. Heterodyning or Doppler spectroscopy (DS) is commonly employed in the microwave region. It has a number of advantages compared with IFS including the ability to overcome certain types of noise, and the fact that the Doppler spectrum is directly proportional to the spectrum of the incident field whatever its statistical properties. Moreover, Pike (1970) has pointed out that under certain conditions the statistical accuracy of measurements may be improved by this technique.

In this letter we report the results of a quantitative analysis of the statistical errors which arise when heterodyne detection is used in conjunction with the post-detection processing technique of digital autocorrelation of clipped photon counting fluctuations which has been reported and discussed fully elsewhere (Jakeman and Pike 1969, Jakeman 1970, Jakeman *et al* 1971). We set aside the first two advantages of DS mentioned above and compare only the statistical accuracy of equivalent IFS and heterodyne measurements. Using the notation of earlier papers, the normalized intensity autocorrelation function of a heterodyned signal in the short sample-time limit takes the form

$$g^{(2)}(\tau) = 1+2\frac{I_s I_c}{\bar{I}^2}|g^{(1)}(\tau)|\cos\omega\tau+\left(\frac{I_s}{\bar{I}}\right)^2(g_s{}^{(2)}(\tau)-1) \qquad (1)$$

where the mean intensity $\bar{I}$ is equal to the sum of the signal component I_s and the coherent contribution I_c, $g_s{}^{(2)}(\tau)$ is the signal intensity correlation function and $g^{(1)}(\tau)$ is the Fourier transform of the optical spectrum assumed to be symmetric and centred at a frequency $\omega = \omega_c - \omega_s$ from the coherent component. The second or Doppler term in equation (1) is proportional to the desired optical spectrum whatever the statistics of the incident field. The final or IFS term in this equation complicates the form of $g^{(2)}(\tau)$, but its effect can be minimized by ensuring that $I_c \gg I_s$. Removal of the analogous term in the single clipped case is not so easily accomplished. The normalized autocorrelation function of photon-counting fluctuations clipped at k in one channel is defined by

$$g_k^{(2)}(\tau) = \langle n_k(0)n(\tau)\rangle / \bar{n}_k\bar{n}$$

with

$$n_k(t)\begin{cases} = 1 & n(t) > k \\ = 0 & n(t) \leqslant k \end{cases}$$

$n(t)$ being the number of photoelectrons counted in the sample time T at time t. In the case of a heterodyned gaussian signal, for example, it may be written in the form

$$g_k^{(2)}(\tau) = 1 + \frac{2\bar{n}_s\bar{n}_c}{\bar{n}\bar{n}_k}|g^{(1)}(\tau)|\cos\omega t\frac{\partial\bar{n}_k}{\partial\bar{n}_c} + \frac{\bar{n}_s^2|g^{(1)}(\tau)|^2}{\bar{n}\bar{n}_k}\frac{\partial}{\partial\bar{n}_c}\left(\bar{n}_c\frac{\partial\bar{n}_k}{\partial\bar{n}_c}\right) \tag{2}$$

where the mean clipped count rate is given by

$$\bar{n}_k = 1 - \exp\left(-\frac{\bar{n}_c}{1+\bar{n}_s}\right)\left\{e_k\left(\frac{\bar{n}_c}{(1+\bar{n}_s)^2}\right) - \left(\frac{\bar{n}_s}{1+\bar{n}_s}\right)^{k+1} e_k\left(\frac{\bar{n}_c}{(1+\bar{n}_s)\bar{n}_s}\right)\right\} \tag{3}$$

with

$$e_k^{(x)} = \sum_{n=0}^{k}\frac{x^n}{n!}$$

and $\bar{n}$, $\bar{n}_s$, $\bar{n}_c$ are mean photoelectron counting rates. Equation (2) is an exact result which follows from the generating function for the heterodyne case (Jakeman 1970) and the formula for the single clipped autocorrelation function given in the original paper on clipping by Jakeman and Pike (1969). Calculations show that the ratio R_k of the 'IFS' to the 'Doppler' terms is given by

$$R_k \simeq \tfrac{1}{2}\frac{\bar{n}_s}{\bar{n}_c} \qquad \bar{n}_s \ll 1$$

$$\left.\begin{aligned} R_0 &= \tfrac{1}{2}\frac{n_s}{n_c}\left(1 - \frac{\bar{n}_c}{1+\bar{n}_s}\right) \\ R_{k\gg 1} &= \tfrac{1}{2}\frac{\bar{n}_s}{\bar{n}_c}\left(1 - \frac{\bar{n}_s\bar{n}_c}{(1+\bar{n}_s)^2}\right) \end{aligned}\right\} \qquad \bar{n}_s \gtrsim 1.$$

Only when $\bar{n}_s \ll 1$ can the IFS term be minimized as in the unclipped case by ensuring that $\bar{n}_c \gg \bar{n}_s$. If $\bar{n}_s > 1$ and $1+\bar{n}_s < \bar{n}_c < (1+\bar{n}_s)^2/\bar{n}_s$, R_k changes sign as k increases from zero and can therefore again be made small by an appropriate choice of k. Attainment of values of $\bar{n}_s$, $\bar{n}_c$ and k fulfilling these requirements will be difficult in practice and since the advantages of heterodyning may only justify its use in the weak signal limit, the remainder of the letter will be confined to analysis of the case $\bar{n}_s \ll 1$, $\bar{n}_s \ll \bar{n}_c$.

When the intensity of the signal to be analysed is sufficiently low the major contribution to the noise in the detector output is the shot noise due to the coherent component. The statistical accuracy calculations presented in an earlier paper (Jakeman *et al* 1971) then simplify considerably. For an ideal correlator, after biassing has been taken into account we obtain the result:

$$\frac{\operatorname{Var}\hat{g}^{(2)}(\tau)}{(g^{(2)}(\tau)-1)^2} = (4N\bar{n}_s{}^2|g^{(1)}(\tau)|^2\cos^2\omega\tau)^{-1} \qquad \text{(DS)} \tag{4}$$

where the estimator for the intesity autocorrelation function $g^{(2)}(\tau)$ is defined by

$$\hat{g}^{(2)}(\tau) = \frac{1}{N}\sum_i^N n(t_i)n(t_i+\tau)\bigg/\frac{1}{N^2}\left(\sum_i^N n(t_i)\right)^2 \tag{5}$$

and N is the total number of samples. From previous results (Jakeman *et al* 1971) in the absence of the coherent component (IFS case)

$$\frac{\operatorname{Var}\hat{g}^{(2)}(\tau)}{(g^{(2)}(\tau)-1)^2} = \frac{1+|g^{(1)}(\tau)|^2}{N\bar{n}_s{}^2|g^{(1)}(\tau)|^4} \qquad \text{(IFS)}. \tag{6}$$

The ratio of expressions (4) and (6) ranges from $\frac{1}{8}$ at $\tau = 0$ to zero as $\tau \to \infty$ indicating an advantage in heterodyning of about one order of magnitude. As an example consider the accuracy of a measurement of the linewidth Γ of a lorentzian spectral feature using a parallel channel ideal correlator. Assuming $\omega = 0$ for simplicity (homodyne case) a two parameter fitting procedure using results (4) and (6) leads, in the limit of a large number of autocorrelator channels, to

$$\left(\frac{\operatorname{Var}\hat{\Gamma}}{\Gamma^2}\right)_{\text{ideal}} = \begin{cases} 2\Gamma T/\bar{n}_s{}^2 N & \text{DS} \qquad (7)\\ 21\cdot 2\Gamma T/\bar{n}_s{}^2 N & \text{IFS}. \qquad (8)\end{cases}$$

A factor 10 is therefore gained by homodyning with a coherent reference beam.

In the case of autocorrelation of the single clipped homodyned signal it is not difficult to show that

$$\operatorname{Var}\hat{g}_k{}^{(2)}(\tau) = \frac{1}{N\bar{n}}\left(\frac{1}{\bar{n}_k}-1\right). \tag{9}$$

$\hat{g}_k{}^{(2)}(\tau)$, defined in analogy with equation (5), is the estimator of the single clipped intensity autocorrelation function which in the present approximation is given by

$$g_k{}^{(2)}(\tau) \simeq 1+2\bar{n}_s\bar{n}_c{}^k\exp(-\bar{n}_c)|g^{(1)}(\tau)|/\gamma(1+k,\bar{n}_c) \tag{10}$$

where $\gamma(\alpha,\beta)$ is the incomplete gamma function. Equation (10) with incorrect normalization appeared in a previous paper by the present author (Jakeman 1970). For a lorentzian spectrum the two-parameter fitting procedure using results (9) and (10) leads to

$$\left(\frac{\operatorname{Var}\hat{\Gamma}}{\Gamma^2}\right)_{\substack{\text{single}\\ \text{clipped}}} = E(k,\bar{n}_c)\left(\frac{\operatorname{Var}\hat{\Gamma}}{\Gamma^2}\right)_{\text{ideal}} \qquad \text{(DS)} \tag{11}$$

where

$$E(k,\bar{n}_c) = \frac{k!}{\bar{n}_c{}^{2k+1}}e_k(\bar{n}_c)\{\exp(\bar{n}_c)-e_k(\bar{n}_c)\}. \tag{12}$$

When both channels are clipped prior to autocorrelation the following results are obtained:

$$\mathrm{Var}\, \hat{g}_{kk}^{(2)}(\tau) = (1-\bar{n}_k)^2/N\bar{n}_k^2 \tag{13}$$

$$g_{kk}^{(2)}(\tau) \simeq 1+2\bar{n}_s\bar{n}_c^{2k+1}\exp(-2\bar{n}_c)|g^{(1)}(\tau)|/\gamma^2(1+k, n_c) \tag{14}$$

$$\left(\frac{\mathrm{Var}\,\hat{\Gamma}}{\Gamma^2}\right)_{\substack{\text{double}\\ \text{clipped}}} = E^2(k,\bar{n}_c)\left(\frac{\mathrm{Var}\,\hat{\Gamma}}{\Gamma^2}\right)_{\text{ideal}} \qquad (\text{DS}). \tag{15}$$

The function $E(k, n_c)$ has a minimum for $k \simeq n_c$ which increases as these quantities increase until the asymptotic value $\pi/2$ is approached for $k \simeq n_c \gg 1$. Thus, at worst, clipping increases the standard deviation of Γ by a factor 1·5 over the ideal homodyne case. Since in the absence of the coherent component (IFS) the optimum clipping level is zero for $n_s \ll 1$ and would lead to the result (8), it is evident that a factor of at least 7 can be gained in experiment time by using Doppler rather than intensity fluctuation spectroscopy. This is because in DS information is carried by a cross-product of the signal with the less noisy coherent component (third term of equation (1)) rather than by the square of the signal as in the IFS case.

It is not difficult to take finite sample time effects into account in the above calculations. For a lorentzian spectrum and with a finite number of autocorrelator channels, the optimum choice of T is such that $\Gamma T \simeq 0{\cdot}1$ as in the IFS case. The main conclusions of the letter are thus not affected. This may not be true for spatial coherence effects due to the presence of finite apertures in the system and these have yet to be investigated.

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